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On the upper chromatic numbers of the reals

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Abstract

Let S be a metric space and let k be a positive integer. Define $\hat{\chi}^{(k)}(S)$ to be the smallest positive integer m such that for every $k \times m$ array $D=(D_{ij})$ of positive real numbers, S can be colored with the colors $1,2,\dots,m$ such that no two points of distance D_{ij} are both colored j. We improve the best upper bound known on $\hat{\chi}^{(k)}(\mathbb{R})$ from $32^k k!$ to $\lceil 4ek \rceil$, where e is the base of the natural logarithm. We prove a conjecture of Abrams (Discrete Math. 169 (1997) 157–162) that $\hat{\chi}^{(k)}(\mathbb{Z}) = \hat{\chi}^{(k)}(\mathbb{R})$ for all $k \in \mathbb{N}$, extend this result to higher dimensions under the l^1 and l^∞ norms, and prove that the upper chromatic numbers are finite for these spaces. We also introduce a new related chromatic quantity of a graph G, the *chromatic capacity*, $\chi_{\text{cap}}(G)$.
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1. Introduction

Let S be a metric space with distance function d, let $k,m \in \mathbb{N}$, let $D=(D_{ij})$ be a $k \times m$ array of positive real numbers, and let D_j denote the set of numbers in the jth column of D. As usual, $[m]$ denotes the set $\{1,2,\dots,m\}$. A coloring $C: S \to [m]$ is a *D-coloring* of S if for all $x,y \in S$ such that $C(x)=C(y)=j$, we have $d(x,y) \notin D_j$. If a D-coloring of S exists, then S is *D-colorable*. Each element of D_j is a *restriction* for the color j, and D is a *restriction array*.

Definition. The kth *upper chromatic number* of S, written $\hat{\chi}^{(k)}(S)$, is the smallest positive integer m such that S is D-colorable for every $k \times m$ restriction array D. If no such integer exists, we write $\hat{\chi}^{(k)}(S)=\infty$.

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For $\hat{\chi}^{(1)}(S)$ we write $\hat{\chi}(S)$, and call this the *upper chromatic number* of S. Observe that $\hat{\chi}^{(k)}(S)$ is increasing in both k and S.

In [9] Greenwell and Johnson introduce the upper chromatic number for Euclidean spaces and prove two results. First, $\hat{\chi}(\mathbb{R})=3$. Second, if there exist $r>0$ and $l\in\mathbb{N}$ such that $\mathbb{R}^n$ is D-colorable for all $1\times l$ restriction arrays D such that $D_{i+1}\geqslant rD_i$ for $i=1,\ldots,(l-1)$, then $\hat{\chi}(\mathbb{R}^n)<\infty$. They also pose the still-open question of whether $\hat{\chi}(\mathbb{R}^2)$ is finite. Observe that $\hat{\chi}(S)$ is a generalization of the ordinary chromatic number of Euclidean space, $\chi(\mathbb{R}^n)$, where only arrays of 1's are considered. In 1961, Hadwiger [10] proved that $4\leqslant\chi(\mathbb{R}^2)\leqslant 7$, and these are still the best bounds known for $\chi(\mathbb{R}^2)$ (see [7, pp. 841–842]).

In [1] Abrams proves that $\hat{\chi}^{(k)}(\mathbb{R})$ is finite for all k, and achieves an upper bound of $32^k k!$. We improve this bound to $\lceil 4ek\rceil$, where e is the base of the natural logarithm (see Corollary 6). He also proves $\hat{\chi}^{(2)}(\mathbb{Z})\geqslant 4$, and conjectures $\hat{\chi}^{(k)}(\mathbb{Z})=\hat{\chi}^{(k)}(\mathbb{R})$ for all k. We prove this conjecture in Theorem 8.

In Section 2 we apply the compactness principle to reduce our problem to that of coloring finite sets. In Section 3 we interpret $\hat{\chi}^{(k)}(S)$ in terms of graph theory, prove a sufficient condition for an edge-colored multigraph with bounded degree to have a compatible vertex coloring, and obtain our linear bound on $\hat{\chi}^{(k)}(\mathbb{R})$ as a consequence of this result. In Section 4 we prove that $\hat{\chi}^{(k)}(\mathbb{Z})=\hat{\chi}^{(k)}(\mathbb{R})$ for all k, generalize this result to $\mathbb{Z}^n$ and $\mathbb{R}^n$ under the l^1 and l^∞ norms and demonstrate that even apparently sparse subsets A of $\mathbb{Z}$ can have $\hat{\chi}^{(k)}(A)=\hat{\chi}^{(k)}(\mathbb{R})$ for all k. We conclude the section by showing $\hat{\chi}^{(k)}(\mathbb{R}^n)$ is finite under both the l^1 and the l^∞ norm. In Section 5 we offer some comments on open problems concerning $\hat{\chi}^{(k)}(\mathbb{Z})$ and $\hat{\chi}(\mathbb{R}^2)$ (under the Euclidean metric), and introduce the chromatic capacity of a graph, which is closely related to the concept of m-emulsivity developed independently of us in [4,3,5].

2. Compactness

By the *compactness principle*, we need focus only on coloring all finite subsets of S.

Theorem 1 (Compactness Principle [13, in proof of Theorem 2.2, p. 1800]). *Let S be an infinite set and suppose $\mathcal{U}$ is a family of pairs (A,C) such that $A\subseteq S$ and C is an m-coloring of A; that is $C: A\to[m]$. Suppose $\mathcal{U}$ is closed under restriction; that is, if $(A,C)\in\mathcal{U}$ and $B\subseteq A$ then $(B,C|_B)\in\mathcal{U}$. Suppose also that for all finite $A\subseteq S$ there is a C with $(A,C)\in\mathcal{U}$. Then there exists a $C: S\to[m]$ such that for all finite $A\subseteq S$ we have $(A,C|_A)\in\mathcal{U}$.*

In our case, we fix a restriction array D and let

$$\mathcal{U}=\{(A,C)\colon A\subseteq S,\ C \text{ is a } D\text{-coloring of } A\}.$$

Then Theorem 1 yields the following result.

Proposition 2. *If all finite subsets $F \subseteq S$ are D-colorable, then S is D-colorable.*

Proposition 2 allows us to apply probabilistic methods. It also implies, by a scaling argument, that $\hat{\chi}^{(k)}(\mathbb{R}^n) = \hat{\chi}^{(k)}(S)$ for any subset S of $\mathbb{R}^n$ that contains a ball.

3. Graph–theoretic formulation and linear bound on $\hat{\chi}^{(k)}(\mathbb{R})$

We reformulate the problem in terms of *edge-colored* multigraphs.

Definition. Given $F \subseteq S$ and a $k \times m$ restriction array D, let $G(F,D) = (V,E)$ be the *edge-colored multigraph* with vertex set $V = F$, and edge set

$$E = \{(\{x,y\},j)\colon x,y \in F,\ x \neq y, \text{ and } d(x,y) \in D_j\}.$$

An *edge m-coloring* of a multigraph is a coloring of its edges with the set $[m]$.

We consider each edge of $G(F,D)$ to be colored by its second coordinate. This edge-coloring encodes all information about the restrictions D places on the coloring of F.

Definition. Given an edge m-colored multigraph G, a *compatible vertex coloring* of G is a coloring of the vertices of G with the set $[m]$ such that no edge is colored the same as both its vertices.

Observe that a D-coloring of F corresponds to a compatible vertex coloring of $G(F,D)$.

Our first theorem on compatible vertex colorings of edge-colored multigraphs uses the Lovász local lemma (see [2, pp. 53–55]), which we state here for completeness. In the lemma, $\Pr(A)$ denotes the probability of the event A, and e is the base of the natural logarithm.

Definition. The directed graph $H = (V,E)$ is a *dependency digraph* on the events $A_1,\ldots,A_n$ of a probability space if $V = \{A_1,\ldots,A_n\}$ and for all $i \in [n]$, the event A_i is independent of all boolean combinations of the events $\{A_j\colon j \neq i, (A_i,A_j) \notin E\}$. (Note that a set of events may have more than one dependency digraph.) For fixed H, the maximum vertex sub-degree of H is a *dependency bound* for $\{A_1,\ldots,A_n\}$.

Theorem 3 (Lovász local lemma). *Let Δ be a dependency bound for the events $\{A_1,\ldots A_n\}$. If $\Pr(A_i) \leqslant 1/e(\Delta+1)$ for all $i \in [n]$, then $\Pr(\cap A_i^c) > 0$.*

Theorem 4. *If G is an edge m-colored multigraph with maximum vertex degree M, and $m^2 \geqslant e(2M-1)$, then G has a compatible vertex coloring.*

Proof. By compactness, it is sufficient to color all finite subgraphs $H=(V,E)$ of G. We color each vertex of H randomly and independently with the color set $[m]$, each color having probability $1/m$. For each $f=(\{x,y\},j)\in E$, let A_f be the event that x and y are both colored j. Then $2M-2$ is a dependency bound for these events, since A_f depends only on events A_g where f and g share a vertex. But

$$\Pr(A_f)=\frac{1}{m^2}\leqslant\frac{1}{e(2M-1)}$$

for all $f\in E$, so a compatible vertex coloring exists by the Lovász local lemma. □

Corollary 5. *Let $N\in\mathbb{N}$. If S is a metric space such that for all $x\in S$ and $r>0$ we have $|\{y\colon d(x,y)=r\}|\leqslant N$, then $\hat{\chi}^{(k)}(S)\leqslant\lceil 2eNk\rceil$.*

Proof. If the restriction array D has size $k\times m$, then there are no more than km restricted distances, so $G(S,D)$ has maximum vertex degree Nmk, and we can apply Theorem 4. □

Corollary 6. $\hat{\chi}^{(k)}(\mathbb{R})\leqslant\lceil 4ek\rceil$.

Proof. We apply Corollary 5 with $N=2$. □

4. Integer lattices with the l^1 and l^∞ norms

In [1] Abrams conjectures $\hat{\chi}^{(k)}(\mathbb{Z})=\hat{\chi}^{(k)}(\mathbb{R})$ for all k. We prove this is true in general for the integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ when endowed with either the l^1 norm

$$\|x-y\|_1=\sum_{t=1}^{n}|x_t-y_t|$$

or l^∞ norm

$$\|x-y\|_\infty=\max_t|x_t-y_t|$$

(where x_t denotes the tth coordinate of the n-vector x). The proofs rely on the following fact about systems of linear equations.

Lemma 7. *Let $\mathscr{S}$ be a finite system of homogeneous linear equations with rational coefficients, i.e. $Ax=0$ for some rational matrix A. If $\hat{x}\in\mathbb{R}^n$ is a real solution to $\mathscr{S}$, then there exist rational solutions $x'\in\mathbb{Q}^n$ arbitrarily close to $\hat{x}$.*

The lemma follows from the solution space having a basis of rational vectors.

Theorem 8. *Under the l^1 norm, $\hat{\chi}^{(k)}(\mathbb{Z}^n)=\hat{\chi}^{(k)}(\mathbb{R}^n)$ for all $k,n\in\mathbb{N}$.*

Proof. $\hat{\chi}^{(k)}(\mathbb{Z}^n)\leqslant\hat{\chi}^{(k)}(\mathbb{R}^n)$ is trivial from the subset relation. We now show the reverse inequality. Let $m<\hat{\chi}^{(k)}(\mathbb{R}^n)$. Then there exist a $k\times m$ array $\hat{D}$ and finite set $\hat{F}=\{\hat{x}^1,\ldots,\hat{x}^N\}\subseteq\mathbb{R}^n$ such that $G(\hat{F},\hat{D})$ has no compatible vertex coloring. We construct a $k\times m$ array D' of positive rationals and a finite set $F'\subseteq\mathbb{Q}^n$ such that $G(F',D')$ contains an isomorphic copy of $G(\hat{F},\hat{D})$, so that F' is not D'-colorable. Scaling F' and D' to clear denominators yields a restriction array D'' and an $F''\subseteq\mathbb{Z}^n$ that is not D''-colorable.

Let

$$s_{pqt}=\operatorname{sign}(\hat{x}_t^p-\hat{x}_t^q)=\begin{cases}1 & \text{if } \hat{x}_t^p\geqslant\hat{x}_t^q,\\ -1 & \text{otherwise.}\end{cases}$$

We define $\mathscr{S}$, a system of linear equations in the $mk+nN$ variables D_{ij} (where $i=1,2,\ldots,k$ and $j=1,2,\ldots,m$), and x_t^p (where $p=1,2,\ldots,N$ and $t=1,2,\ldots,n$). Let $\mathscr{S}$ include one equation

$$x_t^p-x_t^q=0$$

for each p,q,t such that $\hat{x}_t^p=\hat{x}_t^q$, and one equation

$$\sum_{t=1}^{n}s_{pqt}(x_t^p-x_t^q)-D_{ij}=0 \tag{1}$$

for each p,q,i,j such that $\|\hat{x}^p,\hat{x}^q\|_1=\hat{D}_{ij}$. In other words, each edge $(\{\hat{x}^p,\hat{x}^q\},j)$ in $G(\hat{F},\hat{D})$ generates an equation of form (1). The system $\mathscr{S}$ has a non-trivial real solution (namely $x_t^p=\hat{x}_t^p$, $D_{ij}=\hat{D}_{ij}$), and the coefficients of the variables in $\mathscr{S}$ are all 0 or ±1, so we can apply Lemma 7 to obtain a rational solution $F'=\{x^{1'},\ldots,x^{N'}\}$, $D'=(D_{ij'})$ arbitrarily close to $\hat{F}$, $\hat{D}$.

By taking F' and D' close enough to $\hat{F}$ and $\hat{D}$ so that the N points of F' are distinct, $D'>0$ and each choice of sign remains correct in (1) when removing the absolute value from the l^1 distance expression, we ensure that $G(F',D')$ contains an isomorphic copy of $G(\hat{F},\hat{D})$. Scaling both F' and D' yields an appropriate integer solution. $\square$

Theorem 9. *Under the l^∞ norm, $\hat{\chi}^{(k)}(\mathbb{Z}^n)=\hat{\chi}^{(k)}(\mathbb{R}^n)$ for all $k,n\in\mathbb{N}$.*

Proof. The proof is similar to that of Theorem 8, but with a different system $\mathscr{S}$ of equations. Let s_{pqt} be as before, but this time let

$$\begin{aligned}T_{pq}&=\{t\colon |\hat{x}_t^p-\hat{x}_t^q|=\|\hat{x}^p-\hat{x}^q\|_\infty\}\\ &=\{t\colon t \text{ maximizes } |\hat{x}_t^p-\hat{x}_t^q|\}\end{aligned}$$

and let $\mathscr{S}$ include one equation

$$s_{pqt}(x_t^p-x_t^q)-D_{ij}=0$$

for each p,q,t,i,j such that $\|\hat{x}^p-\hat{x}^q\|_\infty=\hat{D}_{ij}$ and $t\in T_{pq}$.

By choosing our rational solution sufficiently close to the real solution to guarantee the points in F' are distinct, $D'>0$, and

$$|(x_t^p)'-(x_t^q)'|\geqslant|(x_s^p)'-(x_s^q)'|$$

for all p,q and all $t\in T_{pq}$, $s\notin T_{pq}$, we obtain that $G(F',D')$ contains an isomorphic copy of $G(\hat{F},\hat{D})$. $\square$

Note that neither of these proofs rely on the finiteness of $\hat{\chi}^{(k)}(\mathbb{R}^n)$, although we do prove finiteness at the end of this section. Both of these theorems prove Abrams' conjecture when $n=1$, and by applying compactness we get slightly more.

Corollary 10. *If A is a subset of the reals that contains arithmetic progressions of arbitrary length, then $\hat{\chi}^{(k)}(A)=\hat{\chi}^{(k)}(\mathbb{R})$ for all $k\in\mathbb{N}$.*

Proof. By Theorem 8, it suffices to show $\hat{\chi}^{(k)}(A)=\hat{\chi}^{(k)}(\mathbb{Z})$. Let $m=\hat{\chi}^{(k)}(\mathbb{Z})-1$. By compactness there exist $l\in\mathbb{N}$ and a $k\times m$ restriction array D such that $[l]$ is not D-colorable. The set A contains an arithmetic progression P of length l and common difference x. But P is not xD-colorable, where xD is the array obtained by multiplying each element of D by x. Hence,

$$\hat{\chi}^{(k)}(A)\geqslant\hat{\chi}^{(k)}(P)\geqslant m+1=\hat{\chi}^{(k)}(\mathbb{Z})\geqslant\hat{\chi}^{(k)}(A).\qquad\square$$

In particular, the Van der Waerden theorem (see [8, pp. 29–30]) asserts that given any finite partition $P_1,\dots,P_n$ of the integers, at least one of the partition classes contains arbitrarily long arithmetic progressions. Szemerédi [14] proved a stronger theorem implying Van der Waerden's result (see also [8, pp. 45–46]). If A is a set of positive upper density, that is,

$$\limsup_{n\to\infty}\frac{|A\cap[n]|}{n}>0,$$

then A contains arbitrarily long arithmetic progressions. The point is that there are arbitrarily sparse subsets of $\mathbb{Z}$ which nonetheless have the same kth upper chromatic number as $\mathbb{R}$, for all k.

The next two theorems show that $\hat{\chi}^{(k)}(\mathbb{R}^n)$ is finite for all $n,k\in\mathbb{N}$, when endowed with the l^1 or l^∞ norm.

Theorem 11. *Under the l^∞ norm, for all $k,n\in\mathbb{N}$, $\hat{\chi}^{(k)}(\mathbb{R}^n)$ is finite and $\hat{\chi}^{(k)}(\mathbb{R}^{n+1})\leqslant m\,\hat{\chi}^{(mk)}(\mathbb{R}^n)$, where $m=\hat{\chi}^{(k)}(\mathbb{R})$.*

Proof. Since $\hat{\chi}^{(k)}(\mathbb{R})$ is finite for all k, we can assume by induction that $\hat{\chi}^{(mk)}(\mathbb{R}^n)$ is finite. Let D be any $k\times m\,\hat{\chi}^{(mk)}(\mathbb{R}^n)$ restriction array, and let (i,j) represent column $(m-1)i+j$ of D (for $i=1,2,\dots,\hat{\chi}^{(mk)}(\mathbb{R}^n)$ and $j=1,2,\dots,m$). Let $D^{(i)}=[D_{(i,1)},\dots,D_{(i,m)}]$. That is, $D^{(i)}$ is a subarray of D, with $D^{(i)}_j=D_{(i,j)}$. Let D' be the $mk\times\hat{\chi}^{(mk)}(\mathbb{R}^n)$ restriction array whose ith column consists of all the entries of $D^{(i)}$. Then we can D'-color $\mathbb{R}^n$ by a coloring C_0. For each i, we can $D^{(i)}$-color $\mathbb{R}$ by a coloring C_i using the colors $(i,1),\dots,(i,m)$. For $x=(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}$, we color x by

$$C(x)=C_{C_0(x_1,\dots,x_n)}(x_{n+1}).$$

We claim C is a D-coloring of $\mathbb{R}^{n+1}$. Let $x,y\in\mathbb{R}^{n+1}$, and suppose $C(x)=C(y)=(i,j)$. Then $C_0(x_1,\dots,x_n)=C_0(y_1,\dots,y_n)=i$. Hence

$$\max_{1\leqslant t\leqslant n}|x_t-y_t|\notin D'_i\supseteq D_{(i,j)},$$

since C_0 is a D'-coloring of $\mathbb{R}^n$. Moreover,

$$|x_{n+1} - y_{n+1}| \notin D_j^{(i)} = D_{(i,j)},$$

since C_i is a $D^{(i)}$-coloring of $\mathbb{R}$. So $\|x - y\|_\infty = \max_{1 \leqslant t \leqslant n+1} |x_t - y_t| \notin D_{(i,j)}$. □

In the next theorem, we write $\mathbb{R}_1^n$ or $\mathbb{R}_\infty^n$ to indicate we are using the l^1 or l^∞ norm, respectively.

Theorem 12. *$\hat{\chi}^{(k)}(\mathbb{R}_1^n) \leqslant \hat{\chi}^{(k)}(\mathbb{R}_\infty^{2^{n-1}})$ for all $k, n \in \mathbb{N}$.*

Proof. Let D be a $k \times \hat{\chi}^{(k)}(\mathbb{R}_\infty^{2^{n-1}})$ restriction array, and let C be a D-coloring of $\mathbb{R}_\infty^{2^{n-1}}$. Let $I \subseteq 2^{[n]}$ (the power set of $[n]$) such that for each $S \subseteq [n]$, exactly one of S and $[n] \setminus S$ is in I, and index the coordinates of $\mathbb{R}^{2^{n-1}}$ by I. We define a function $f : \mathbb{R}_1^n \to \mathbb{R}_\infty^{2^{n-1}}$ as follows: for $S \in I$, let the Sth coordinate of $f(x_1, \ldots, x_n)$ be

$$f_S(x_1, \ldots, x_n) = \sum_{i \in S} x_i - \sum_{i \notin S} x_i.$$

Then for all $x, y \in \mathbb{R}^n$, $\| x \|_1 = \| f(x) \|_\infty$, and $f(x - y) = f(x) - f(y)$. So if $C(f(x)) = C(f(y)) = i$ then

$$\|x - y\|_1 = \| f(x - y)\|_\infty = \| f(x) - f(y)\|_\infty \notin D_i.$$

Thus $C \circ f$ is a D-coloring of $\mathbb{R}_1^n$. □

5. Further questions

5.1. Coloring the integers

Theorem 8 increases our interest in calculating $\hat{\chi}^{(k)}(\mathbb{Z})$. The only known value is $\hat{\chi}(\mathbb{Z}) = 3$. In general $\hat{\chi}^{(k)}(\mathbb{Z}) > k$, since no $(k+1)$ consecutive integers are D-colorable if D is the $k \times k$ restriction array whose columns each contain the integers $1, \ldots, k$. Since for

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix},$$

$\mathbb{Z}$ is not D-colorable (see [1]), $\hat{\chi}^{(k)}(\mathbb{Z}) \geqslant 4$. Abrams [1] suggests that indeed $\hat{\chi}^{(2)}(\mathbb{Z}) = 4$, but the best upper bound proved is $\hat{\chi}^{(2)}(\mathbb{Z}) \leqslant 22$ from Corollary 6. Hence we pose:

Problem 1. Improve the bounds $4 \leqslant \hat{\chi}^{(2)}(\mathbb{Z}) \leqslant 22$ and, more generally, $k + 1 \leqslant \hat{\chi}^{(k)}(\mathbb{Z}) \leqslant \lceil 4ek \rceil$.

Problem 2. Improve the bounds on $\hat{\chi}^{(k)}(\mathbb{Z}_\infty^n)$ and $\hat{\chi}^{(k)}(\mathbb{Z}_1^n)$ given by Theorems 11 and 12.

In searching for restriction arrays to prove lower bounds, it helps to understand D-colorability for smaller arrays D, so the following proposition may be useful.

Proposition 13. *Let* $d = \gcd(x, y, z, t)$. *If*

$$D = \begin{pmatrix} x & z \\ y & t \end{pmatrix},$$

then $\mathbb{Z}$ *is D-colorable if and only if the entries of* $(1/d)D$ *are all odd.*

Proof. $\mathbb{Z}$ is D-colorable if and only if $d\mathbb{Z} = \{dn\colon n \in \mathbb{Z}\}$ is D-colorable, because each congruence class modulo d may be D-colored independently of the others. But $d\mathbb{Z}$ is D-colorable if and only if $\mathbb{Z}$ is $\frac{1}{d}D$-colorable. So we might as well assume $\gcd(x, y, z, t) = 1$, implying one of the entries of D, say x, is odd. If y, z and t are also all odd, then coloring the even integers 1 and the odd integers 2 yields a D-coloring. So we now assume at least one of y, z, and t is even.

Given any D-coloring of $\mathbb{Z}$, assume 0 is colored 1. Then $\pm x$ and $\pm y$ must be colored 2, and similarly

$$r_1 + r_2 + \cdots + r_n, \tag{2}$$

must be colored 2 for odd n and 1 for even n, where

$$r_i \in \begin{cases} \{\pm x, \pm y\} & \text{for } i \text{ odd}, \\ \{\pm z, \pm t\} & \text{for } i \text{ even}. \end{cases}$$

Thus, if $r_1 + r_2 + \cdots + r_n = 0$ for some odd n, we have a contradiction.

For n odd, we can rewrite (2) as

$$(p_x - n_x)x + (p_y - n_y)y + (p_z - n_z)z + (p_t - n_t)t,$$

where p_i is the number of times i appears and n_i is the number of times $(-i)$ appears in (2), for each $i \in \{x, y, z, t\}$, and

$$p_x + n_x + p_y + n_y = 1 + p_z + n_z + p_t + n_t.$$

We can further rewrite this set of sums as the set of all integer linear combinations

$$c_x x + c_y y + c_z z + c_t t \quad \text{where } c_x + c_y + c_z + c_t \text{ is odd.} \tag{3}$$

Since (3) is symmetric in y, z, and t, we may assume y is even. Since x is odd,

$$yx + (-x)y + 0z + 0t = 0$$

is of the form (3), so $\mathbb{Z}$ is not D-colorable. $\square$

5.2. Chromatic capacity

When studying edge-colored multigraphs, a new chromatic quantity suggests itself. We call an edge m-colored multigraph *compatible* if it has a compatible vertex coloring; otherwise we call it *incompatible*. If for every edge m-coloring of G a compatible vertex coloring exists, we say G is *m-compatible*; otherwise, G is *m-incompatible*.

Definition. The *chromatic capacity* of a multigraph G, denoted $\chi_{\text{cap}}(G)$, is the largest m such that G is m-incompatible.

If $\chi(G)$ denotes the usual chromatic number, then clearly

$$\chi_{\text{cap}}(G) \leqslant \chi(G) - 1,$$

because a proper vertex coloring of the graph is compatible with every edge coloring. If every edge has multiplicity $\chi(G) - 1$ then the inequality is tight.

In [6] Erdős gives a probabilistic proof that there exist graphs with arbitrarily large girth and chromatic number. A similar result holds for chromatic capacity.

Theorem 14. *Given any $l, m \in \mathbb{N}$ there exists a graph G with girth$(G) > l$ and $\chi_{\text{cap}}(G) \geqslant m$.*

Theorem 14 follows from a simple modification of the proof of Erdős' result given in [2, p. 35], so we omit the details.

Using Theorem 4 and some elementary properties of finite affine planes (which can be found in [11]), we establish the order of magnitude of $\chi_{\text{cap}}(K_n)$.

Lemma 15. *If there exists a finite affine plane Π of order n, then $\chi_{\text{cap}}(K_{n^2}) \geqslant n - 1$.*

Theorem 16. $(1 - \mathrm{o}(1))\sqrt{n} \leqslant \chi_{\text{cap}}(K_n) \leqslant \lfloor \sqrt{e(2n-3)} \rfloor$.

Lemma 15 gives the lower bound, and Theorem 4 gives the upper bound. Lemma 15 has been obtained independently in a different guise by Cochand and Károlyi [5] for the case where n is a prime and Π is a two-dimensional vector space over a finite field of order n. Since our proof (for general affine planes) is similar to theirs, we omit it. Cochand and Károlyi use the term *m-emulsive* to mean the same thing as m-incompatible. Their motivation for studying this property is a theorem of Rödl [12], that for every directed acyclic graph D there exists a graph G such that every acyclic orientation of G contains an induced subgraph isomorphic to D. Cochand and Duchet [4] showed how to construct such a graph G. They use an m-emulsive graph H in the construction, and the size of G is very sensitive to $\Delta(H)$, the maximum vertex degree of H. Hence, they were interested in finding graphs which, in our terminology, have small maximum degree and large chromatic capacity. In this connection, Brightwell and Kohayakawa [3] proved the upper bound in Theorem 16.

Problem 3. Study the chromatic capacity of other graphs.

Problem 4. There are examples of non-bipartite simple graphs G such that $\chi_{\text{cap}}(G) = \chi(G) - 1$. Do there exist simple graphs G of arbitrarily high chromatic number with this property?

5.3. Is $\hat{\chi}(\mathbb{R}^2)$ finite?

It is still an open question (posed originally in [9]) whether $\hat{\chi}(\mathbb{R}^2)$ is finite (under the Euclidean metric). When trying to prove $\hat{\chi}(\mathbb{R}^2) < \infty$, several observations narrow the field of focus. Let $F \subseteq \mathbb{R}^2$, and D be a $1 \times m$ restriction array. First, by compactness, we need focus only on coloring finite F. Second, we need consider only those F and D such that every vertex of $G(F,D)$ is incident to an edge of each color. Otherwise we could color that vertex with the missing color, and color the rest of the vertices inductively. This is a very restrictive geometric condition. Third, we need consider only those arrays D whose elements form a sharply increasing sequence (by the second result from [9] quoted in Section 1).

In this case $G(F,D)$ is a simple graph. If r is the minimum ratio between consecutive restrictions, then in any cycle in $G(F,D)$, the color representing the largest restriction must either appear at least twice or the cycle must be of size at least $1 + \lceil r \rceil$. Since we have only one restriction per color and our vertices are points in $\mathbb{R}^2$, each monochromatic subgraph of $G(F,D)$ must be free of K_4's and $K_{2,3}$'s. Theorem 14 shows that incompatible edge m-colored graphs with these structural properties exist for arbitrarily large m, which is a necessary condition for $\hat{\chi}(\mathbb{R}^2)$ to be infinite.

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References

[1] A. Abrams, The kth upper chromatic number of the line, Discrete Math. 169 (1997) 157–162.
[2] N. Alon, J.H. Spencer, P. Erdős, The Probabilistic Method, Wiley, New York, 1992.
[3] G.R. Brightwell, Y. Kohayakawa, Ramsey properties of orientations of graphs, Random Struct. Algorithms 4 (1993) 413–428.
[4] M. Cochand, P. Duchet, A few remarks on orientations of graphs and Ramsey theory, in: G. Halász, V.T. Sós (Eds.), Irregularities of Partitions, Algorithms and Combinatorics, Vol. 8, Springer, Berlin, 1989, pp. 39–46.
[5] M. Cochand, G. Károlyi, On a graph colouring problem, Discrete Math. 194 (1999) 249–252.
[6] P. Erdős, Graph theory and probability, Canad. J. Math. 11 (1959) 34–38.
[7] P. Erdős, G. Purdy, Extremal problems in combinatorial geometry, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Vol. 2, North-Holland, Amsterdam, 1995, pp. 809–874.
[8] R.L. Graham, B.L. Rothschild, J.H. Spencer, Ramsey Theory, 2nd Edition, Wiley, New York, 1990.
[9] D. Greenwell, P. Johnson Jr., Forbidding prescribed distances for designated colors, Geombinatorics 2 (1992) 13–16.
[10] H. Hadwiger, Ungelöste Probleme N. 40, Elemente Math. 16 (1961) 103–104.
[11] I.N. Herstein, I. Kaplansky, Matters Mathematical, Harper & Row, New York, 1974, pp. 153–176.

[12] V. Rödl, A generalization of Ramsey theorem, in: Graphs, Hypergraphs and Block Systems, Zielona Gora, 1976, pp. 211–220.
[13] J. Spencer, Probabilistic methods, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Vol. 2, North-Holland, Amsterdam, 1995, pp. 1785–1817.
[14] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta Arith. 27 (1975) 199–245.